Reduced Maxwell-Duffing description of extremely short pulses in nonresonant media

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The propagation of extremely short pulses of an electromagnetic field (electromagnetic spikes) is considered in the framework of a model wherein the material medium is represented by anharmonic oscillators with cubic nonlinearities (Duffing model) and waves can propagate only in the right direction. The system of reduced Maxwell-Duffing equations admits two families of exact analytical solutions in the form of solitary waves. These are bright spikes propagating on a zero background, and bright and dark spikes propagating on a nonzero background. We find that these steady-state pulses are stable in terms of boundedness of the Hamiltonian. Direct simulations demonstrate that these pulses are very robust against perturbations. We find that a high-frequency modulated electromagnetic pulse evolves into a breather-like one. Conversely, a low frequency pulse transforms into a quasiharmonic wave.

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I. INTRODUCTION

Extremely short pulses (ESPs) of the electromagnetic field, which contain a few optical cycles, down to even half a cycle have recently attracted a great deal of attention. Much work has already been done on the theory of interaction and propagation of ESPs assuming a resonant medium [1–15] or a nonresonant medium [16–24]. Surveys can be found in [25–28].

As is customary, the description of the ESP evolution uses the total Maxwell equations without assuming a separation into a carrier wave and an envelope. Generally, the Maxwell equations admit the propagation of electromagnetic waves in both directions. If, however, the nonlinear contribution to the polarization of the medium is small, the *unidirectional wave propagation* may be assumed [2,4–6] (see also [7,10] for the resonant case and [18,19] for the nonresonant one). This approximation reduces the wave equation to a first-order one without any assumption about the shape of the waves. The unidirectional wave propagation approximation is frequently used for the simulation of ESP propagation in a homogeneous low density medium [22,27].

The typical models of the nonlinear medium are the ensembles of N-level atoms or anharmonic oscillators. If the spectral width of the ESP is much smaller than the resonance frequency, we can omit all nonresonant levels and consider only two levels. Thus, we obtain *the approximation of the two-level atoms*, which is very popular in resonant and coherent optics. The wave equation in this case is complemented by the Bloch equations for the two-level atom variables. When the unidirectional wave propagation approximation is taken into account, the total Maxwell-Bloch equations transforms into the *reduced Maxwell-Bloch* (RMB) equations [2–4].

The inverse scattering transform method gives an exact solution of the RMB equations which describes the multiple collision of N solitons with different velocities [3,4]. The Hamiltonian formulation of the RMB equations was considered in [29]. The r-matrix was found and it was shown that the Poisson brackets are not ultralocal, contrary to the McCall-Hahn equation describing the propagation of coherent ultrashort pulses in the slowly varying envelope and phase approximation. However, the RMB equations represent a more general completely integrable Hamiltonian system.

Exact multisoliton solutions of the RMB equations that incorporate the effect of a permanent dipole [9] were found in [10,11]. In particular, these solutions are good examples of unipolar, nonoscillating electromagnetic solitons, commonly referred to as “electromagnetic bubbles.”

The validity of the two-level approximation in the interaction of atoms with few-cycle light pulses was studied in [15] by considering a simple three-level atom model. It was pointed out that even if the transition frequency between the ground state and the third level is far away from the spectrum of the pulse, this additional transition can make the two-level approximation inaccurate. When decreasing the pulse width or increasing the pulse area, the two-level approximation will give rise to non-negligible errors compared with the precise results.

The recent investigation of the propagation of an attosecond pulse in a dense two-level medium [14] shows that the standard area theorem breaks down even for small-area pulses. Ideal self-induced transparency cannot occur even for a 2π pulse while pulses whose area is not an integer multiple of 2π cannot evolve to 2π pulses, as predicted by the standard area theorem. Significantly higher spectral components can occur on all these small-area propagating pulses due to a strong carrier reshaping.

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Solitary wave propagation in the total Maxwell-Bloch equations was discussed in [13] assuming optical subcycle pulses interacting with a dense medium of two-level atoms. A large blueshift in the transmitted pulse and a large redshift in the reflected pulse are predicted using intrapulse four-wave mixing.

It should be pointed out that when the atomic density is such that there are many atoms within a cubic resonance wavelength, the near-dipole-dipole interactions—which lead to local-field corrections—cannot be neglected. Hence, in a dense two-level medium, the Maxwell-Bloch equations should be modified as was done in [30–32] (see also [33]). Thus the generalization of the RMB equations taking into account the many-levels atom and/or the dipole-dipole interactions usually destroys complete integrability. This is the price of the desired generalization. Furthermore, in a strong electromagnetic field the picture of energy levels, related with an initial unperturbated Hamiltonian may be not correct. There are different attempts to describe the nonlinear propagation and interaction of electromagnetic pulses in transparent medium beyond the resonant systems.

One of these approaches is to describe the nonlinear dynamics of the medium driven by the electromagnetic field using anharmonic oscillators. In particular, the propagation of a linearly polarized ESP was considered [21,23] using Duffing oscillators so that the nonlinear response of the medium is cubic. This is the simplest generalization of the Lorentz model which has been very useful to describe the propagation of an electromagnetic wave in a linear medium. Recently, the Lorentz oscillator model was employed [22] to account for a linear retarded response of the medium and a nonlinear oscillator was considered to describe an instantaneous Kerr nonlinearity. The Duffing model takes into account the dispersion properties of both the linear and nonlinear responses of the medium so that it may represent better the nonlinear response on an electromagnetic pulse containing a few cycles.

In some cases the anharmonic oscillator model can be derived from the two-level atoms model. For example, in [27] the RMB equations were transformed into a modified Korteweg–de Vries equation. One can develop the procedure of derivation of the series of complete integrable equations from the RMB. On another hand, we can start from the Heisenberg equations for the operator of the displacement of an electron from its equilibrium position. After calculation of the expected values of this variable, omitting the quantum correlations effects, we can obtain the classical equation of motion for the (strongly) anharmonic oscillator. In this way we do not use the two-level (or N-level) atom representation. The Duffing oscillator is the simplest variant of this model, in which only the weak nonlinear response of a medium is accounted for. This model cannot describe a number of effects seen in a strong field of ESPs; for example, the ionization. More complete representations should be developed using a more realistic potential. The wave equation assuming the unidirectional wave propagation approximation and the Duffing oscillator equation for the medium form a system of equations called the *reduced Maxwell-Duffing* (RMD) equations by analogy with the reduced Maxwell-Bloch equations.

The objective of the present work is to study the unidirectional propagation and interactions of linearly polarized ESPs in a nonlinear dispersive medium modeled by anharmonic oscillators with cubic nonlinearities. The paper is structured as follows. The model is derived in Sec. II. Dynamical invariants or integrals of motion are given in Sec. III. Two families of moving ESP solutions are found analytically and confirmed using Hirota bilinear forms in Sec. IV. In Sec. V the stability of the steady-state solutions of the RMD equations is considered using the variational method. We prove stability of the steady-state ESPs by using the boundedness of the Hamiltonian for a fixed total moment. The propagation of the pulses and their collisions (for both signs of the polarity of the colliding pulses) are investigated numerically in Sec. VI. We conclude in Sec. VII.

II. THE REDUCED MAXWELL-DUFFING MODEL

The one-dimensional propagation of electromagnetic waves in a nonlinear medium is governed by the wave equation

$$\frac{\partial^2 E}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 E}{\partial t^2} = \frac{4\pi}{c^2}\frac{\partial^2 P}{\partial t^2}, \tag{1}$$

where P is the polarization of the medium. According to the unidirectional wave approximation Eq. (1) can be replaced by the first-order equation [6,27]

$$\frac{\partial E}{\partial z} + \frac{1}{c}\frac{\partial E}{\partial t} = -\frac{2\pi}{c}\frac{\partial P}{\partial t}. \tag{2}$$

We adopt a simple anharmonic-oscillator model for the medium, which is commonly used to approximate the medium response for an electromagnetic influence [34] (see also [35]). Here we will consider the oscillator with cubic anharmonicity. In addition, we will assume the case of a homogeneous broadening medium, where all atoms have the same parameters. If X represents the displacement of an electron from its equilibrium position, the equation of motion (which neglects friction) can be written as

$$\frac{\partial^2 X}{\partial t^2} + \omega_0^2 X + \kappa_3 X^3 = \frac{\mathfrak{e}}{m_{eff}}E, \tag{3}$$

where ω_0 is an eigenfrequency of the oscillator, κ_3 is anharmonicity coefficients, $m_{eff}=3m/(\varepsilon+2)$ is the effective mass of the electron. Hereafter, we will use m as a symbol for this effective mass and the symbol $\mathfrak{e}$ denotes the electron charge. Finally, the dynamical variable X is related to the medium polarization, $P=n_A\mathfrak{e}X$, where n_A is the density of oscillators (atoms).

It is suitable to use as independent variables $t=z/l$, $x=\omega_0(t-z/c)$, and to normalize the dependent variables (fields) by

$$e = E/A_0, \quad q = X/X_0, \tag{4}$$

where

$$A_0 = m\omega_0^2 X_0/\mathfrak{e} = m\omega_0^2\mathfrak{e}^{-1}(2\mu/|\kappa_3|)^{1/2}, \quad X_0 = (2\mu\omega_0^2/|\kappa_3|)^{1/2}, \tag{5}$$

$$l^{-1} = 2\pi n_A \mathfrak{e}^2/(mc\omega_0) = \omega_p^2/2c\omega_0, \tag{6}$$

and $\omega_p = (4\pi n_A \mathfrak{e}^2/m)^{1/2}$ is the plasma frequency. In terms of the rescaled variables, Eqs. (2) and (3) take the form

$$\frac{\partial e}{\partial t} = -\frac{\partial q}{\partial x}, \quad \frac{\partial^2 q}{\partial x^2} + q + 2\mu q^3 = e, \tag{7}$$

with the single remaining parameter $2\mu = \kappa_3 X_0^2/\omega_0^2$. The two Eqs. (7) are the final form of the model. In the following we will refer to them as reduced Maxwell-Duffing equations.

III. LAGRANGIAN AND INTEGRALS OF MOTION

The system of RMD equations can be derived as the Euler-Lagrange equations from the action functional

$$S = \int \mathcal{L}[q,\phi]dxdt,$$

where the Lagrangian density is

$$\mathcal{L}[q,\phi] = \frac{1}{2}\frac{\partial\phi}{\partial x}\frac{\partial\phi}{\partial t} + \frac{1}{2}\left(\frac{\partial q}{\partial x}\right)^2 - \frac{1}{2}q^2 - \frac{\mu}{2}q^4 + q\frac{\partial\phi}{\partial x}. \tag{8}$$

Application of the variational procedure to the action S yields equations

$$\frac{\partial^2\phi}{\partial t\partial x} + \frac{\partial q}{\partial x} = 0, \quad \frac{\partial^2 q}{\partial x^2} + q + 2\mu q^3 = \frac{\partial\phi}{\partial x}. \tag{9}$$

Identifying ϕ as a potential for the fields q and e, so that $q \equiv -\partial\phi/\partial t$ and $e = \partial\phi/\partial x$, makes these equations identical to the system of Eqs. (7), which can be further transformed into the single equation

$$\frac{\partial q}{\partial t} + \frac{\partial q}{\partial x} + 6\mu q^2\frac{\partial q}{\partial t} + \frac{\partial^3 q}{\partial t\partial x^2} = 0. \tag{10}$$

From the Lagrangian density (8) we can obtain the density of moments of the fields ϕ and q:

$$\pi_\phi(t,x) = \frac{\partial\mathcal{L}}{\partial\phi_{,t}} = \frac{1}{2}\phi_{,x}(t,x) = \frac{1}{2}e(t,x), \quad \pi_q(t,x) = \frac{\partial\mathcal{L}}{\partial q_{,t}} = 0. \tag{11}$$

The density of the canonical Hamiltonian for this dynamical system can be obtained from $\mathcal{L}$ by means of the standard Legendre transformation

$$\mathcal{H} = \frac{\partial\mathcal{L}}{\partial\phi_{,t}}\phi_{,t} + \frac{\partial\mathcal{L}}{\partial q_{,t}}q_{,t} - \mathcal{L} = -\frac{1}{2}\left(\frac{\partial q}{\partial x}\right)^2 + \frac{1}{2}q^2 + \frac{\mu}{2}q^4 - eq.$$

Thus, the Hamiltonian is

$$H = \int_{-\infty}^{+\infty}\left[-\frac{1}{2}\left(\frac{\partial q}{\partial x}\right)^2 + \frac{1}{2}q^2 + \frac{\mu}{2}q^4 - eq\right]dx. \tag{12}$$

The variable e can be eliminated from it, using RMD equations, so that

$$\mathcal{H} = -\frac{\partial}{\partial x}\left(q\frac{\partial q}{\partial x}\right) + \frac{1}{2}\left(\frac{\partial q}{\partial x}\right)^2 - \frac{1}{2}q^2 - \frac{3\mu}{2}q^4.$$

Omitting the full derivative, the Hamiltonian corresponding to the density (13) takes the form

$$H = \int_{-\infty}^{+\infty}\left[\frac{1}{2}\left(\frac{\partial q}{\partial x}\right)^2 - \frac{1}{2}q^2 - \frac{3\mu}{2}q^4\right]dx. \tag{13}$$

The Hamiltonian is the first integral of motion of the RMD equations. An additional integral of motion is the total moment associated with the field ϕ that one may check on the basis of the RMD equations (7):

$$I_1 = \int_{-\infty}^{+\infty} e(t,x)dx = \int_{-\infty}^{+\infty}\phi_{,x}(t,x)dx = \phi(t,x=\infty) - \phi(t,x=-\infty). \tag{14}$$

The magnitude of this integral is defined by the boundary conditions only, thus, it can be interpreted as a topological charge in the Maxwell-Duffing model.

A third integral can be found by the following. Using the canonical moment one can rewrite Eqs. (7) as

$$\frac{\partial\pi_\phi}{\partial t} = -\frac{1}{2}\frac{\partial q}{\partial x}, \quad \pi_\phi = \frac{1}{2}\left(\frac{\partial^2 q}{\partial x^2} + q + 2\mu q^3\right).$$

From the first equation of this system it follows that

$$\pi_\phi\frac{\partial\pi_\phi}{\partial t} = -\pi_\phi\frac{1}{2}\frac{\partial q}{\partial x}.$$

Taking into account the second equation, one can obtain the expression

$$\frac{\partial\pi_\phi^2}{\partial t} = -\frac{1}{4}\frac{\partial}{\partial x}\left[\left(\frac{\partial q}{\partial x}\right)^2 + q^2 + \mu q^4\right].$$

Thus, one obtains the third integral of motion

$$I_2 = \int_{-\infty}^{+\infty}\pi_\phi^2(t,x)dx = \frac{1}{4}\int_{-\infty}^{+\infty}\left(q + 2\mu q^3 + \frac{\partial^2 q}{\partial x^2}\right)^2 dx. \tag{15}$$

Taking into account the relation (11), this integral may be interpreted as a "pulse energy"

$$4I_2 = \int_{-\infty}^{+\infty} e^2(t,x)dx. \tag{16}$$

It is important to understand the physical contents of these integrals of motion. It should be pointed out that the Lagrangian of the RMD model is an example of a degenerate Lagrangian system. The expressions in (11) indicate that this Lagrangian leads to a constrained Hamiltonian system, where $\pi_\phi(t,x) = (1/2)\phi_{,x}(t,x)$ and $\pi_q(t,x) = 0$ is the primary constraint [36]. The conservation of total moment (14) corresponds to the invariance of the system under consideration with respect to a shift of the field $\phi(t,x)$ by a constant. It is not a space translation symmetry, as it usually occurs, when referring to the moment.

To consider the space-time translation symmetry of the RMD model, it is suitable to denote new variables: $y_1 = t$, $y_2 = x$ and $u_1 = \phi$, $u_2 = q$. For any system, if the space-time variables are not explicitly included in the Lagrangian, there are the conservation laws of the form

$$\sum_{k=1,2} \frac{\partial}{\partial y_k} T_i^k = 0, \tag{17}$$

where the energy-moment tensor T_i^k is denoted as

$$T_i^k = \sum_{a=1,2} \frac{\partial \mathcal{L}}{\partial u_{a,k}} u_{a,i} - \mathcal{L}\delta_{ki}.$$

Here, $u_{a,i} = \partial u_a / \partial y_i$. In the case of the RMD model, we have two integrals of motion resulting from (17) as

$$Q_1 = \int_{-\infty}^{+\infty} T_1^1 dx, \quad Q_2 = \int_{-\infty}^{+\infty} T_2^1 dx. \tag{18}$$

By using the Lagrangian (8) one can find

$$T_1^1 = -\frac{1}{2}\left(\frac{\partial q}{\partial x}\right)^2 + \frac{1}{2}q^2 + \frac{\mu}{2}q^4 - q\frac{\partial \phi}{\partial x}, \quad T_2^1 = -\frac{1}{2}\left(\frac{\partial \phi}{\partial x}\right)^2.$$

The substitution of these expressions into integrals of (18) leads to $Q_1 = H$, $Q_2 = -2I_2$. Thus, we obtain the interpretation integrals I_2 and H as the total moment and total energy, respectively, in the RMD model. Unlike the total canonical moment I_1, the total moment $-2I_2$ reflects the invariance of the RMD model with respect to space translation.

IV. ANALYTICAL SOLUTIONS FOR THE EXTREMELY SHORT PULSES

It seems plausible that the system of RMD equations is not integrable. Nevertheless, some exact analytical solutions, describing the propagation of ESPs without destruction, can be found. In order to obtain these steady-state solutions, one should assume that e and q depend on a single variable, as

$$\eta = x - t/\alpha = \omega_0(t - z/c), \tag{19}$$

with some constant α. An expression for the velocity V of a steadily moving pulse then follows from Eq. (4), given by

$$V^{-1} = c^{-1}[1 + (1/2\alpha)(\omega_p/\omega_0)^2]. \tag{20}$$

Hence, the parameter α defines the velocity of the steady state ESP and we should obtain a one-parametric family of exact analytical solutions of the RMD equations. In general, choosing the boundary conditions results in different solutions of these equations. Here we restrict our attention to solitary wave solutions.

A. Steady state pulse on a zero background

Let us consider the following initial and boundary conditions:

$$e(t = 0, x) = e_0(x),$$

and

$$e_0(x) = 0, \quad q(t,x) = \partial q(t,x)/\partial x = 0, \quad \text{at } x \to \pm\infty. \tag{21}$$

The first equation of the system (7) can be integrated to yield

$$e(t,x) = \alpha q(t,x). \tag{22}$$

Next, the second equation from the system (7) takes the form

$$\frac{d^2 q}{d\eta^2} + (1-\alpha)q + 2\mu q^3 = 0. \tag{23}$$

If $\alpha > 1$ and $\mu > 0$, this equation has a family of exact solutions parametrized by the continuous *positive* parameter $\alpha - 1$ and discrete one, as

$$q(t,x) = \pm\sqrt{(\alpha-1)/\mu}\,\text{sech}[\sqrt{(\alpha-1)}(x - t/\alpha - x_0)], \tag{24}$$

$$e(t,x) = \pm\alpha\sqrt{(\alpha-1)/\mu}\,\text{sech}[\sqrt{(\alpha-1)}(x - t/\alpha - x_0)]. \tag{25}$$

Expression (25) corresponds to the one spike of the electromagnetic field, propagating without form distortion in a nonresonant medium with cubic nonlinearity.

There is an alternative method to obtain the steady-state solution of Eqs. (7), wherein the assumption (19) is not in use. The method follows from the observation that system (7) can be represented in bilinear form by Hirota. If the substitutions

$$e = a/h, \quad q = b/h \tag{26}$$

are used, then Eqs. (7) can be rewritten as

$$D_t(a \cdot h) + D_x(b \cdot h) = 0, \tag{27}$$

$$D_x^2(b \cdot h) = ah - bh, \tag{28}$$

$$D_x^2(h \cdot h) = 2\mu b^2, \tag{29}$$

where Hirota's D-operators $D_x(a \cdot b) = \lim_{x\to x'}(\partial_x - \partial_{x'})a(x)b(x')$ [37], and so on, were used.

Let us make the usual assumption to solve equations (27)–(29) [37,38]:

$$a = \alpha \exp(\theta), \quad b = \beta \exp(\theta), \quad h = 1 + h_1 \exp(\theta) + h_2 \exp(2\theta),$$

where $\theta = kx - \Omega t$. The substitution of these expressions into (27)–(29) results in algebraic equations with respect to $\exp(\theta)$. Equating the coefficients of the different powers of $\exp(\theta)$ to zero, one can obtain the system of equations defining a, b, h_1, and h_2. From (27), we get $\alpha\Omega - \beta k = 0$. From (28), two conditions follow: $h_1 = 0$ and $\beta k^2 = \alpha - \beta$. These expressions lead to the "dispersion relation"

$$\Omega = k/(1 + k^2). \tag{30}$$

Equation (29) yields three relations: $h_1 k^2 = 0$, $4h_2 k^2 = \mu\beta^2$, and $h_1 h_2 k^2 = 0$. As $h_1 = 0$, we have only the second relation defining h_2; i.e., $h_2 = \mu(\beta/2k)^2$. Thus, we find a one-soliton solution of the bilinear equations (27)–(29)

$$a = (1 + k^2)\beta \exp(\theta), \quad b = \beta \exp(\theta),$$
$$h = 1 + \mu(\beta/2k)^2 \exp(2\theta).$$

These relations yield the solution of Eqs. (7), which is correlated with the steady-state one that was obtained earlier:

$$q = \frac{\beta \exp(\theta)}{1 + \mu(\beta/2k)^2 \exp(2\theta)}, \quad e = \frac{(1 + k^2)\beta \exp(\theta)}{1 + \mu(\beta/2k)^2 \exp(2\theta)}. \tag{31}$$

If we introduce the parameter x_0 rather than β according to $\beta=2\mu^{1/2}\exp(kx_0)$, and define $\alpha=1+k^2$, expressions (31) can be written in the form of (24) and (25).

It should be remarked that dispersion relation of this solitons (30) differs from the dispersion relation for linear waves in this model: $\Omega=k/(1-k^2)$.

It would be interesting to find the two-soliton solution of the bilinear equations (27)–(29). However, the standard approach [37,38] is not successful. This may indicate that there are neither two- nor more-soliton solutions of the system of RMD equations.

B. Steady-state pulse on a nonzero background

If the medium is first polarized by a continuum electric field, the oscillator coordinate is shifted from the equilibrium position (i.e., atoms have a constant electronic polarizability induced by the external electric field). Let us denote this new position as q_0. The initial and boundary conditions can be written as

$$e(t=0,x)=e_0(x),$$

and

$$e_0=q_0+2\mu q_0^3,\quad q(t,x)=q_0,\quad \partial q(t,x)/\partial x=0,\quad \text{for } x\to\pm\infty. \tag{32}$$

We introduce the variables $f=q-q_0$ and $u=e-e_0$, which approach zero at infinity. This results in the following equations:

$$\frac{\partial u}{\partial t}=-\frac{\partial f}{\partial x},\quad \frac{\partial^2 f}{\partial x^2}+f+2\mu(3q_0^2 f+3q_0 f^2+f^3)=u. \tag{33}$$

Looking for a steady state solution of (33), we obtain

$$\frac{d^2 f}{d\eta^2}+(1-\alpha_1)f+6\mu q_0 f^2+2\mu f^3=0, \tag{34}$$

where $\alpha_1=\alpha-6\mu q_0^2$. Integrating this equation once, taking into account zero boundary conditions, one obtains

$$\left(\frac{df}{d\eta}\right)^2=(\alpha_1-1)f^2-4\mu q_0 f^3-\mu f^4. \tag{35}$$

The substitution of $f=1/y$ transforms this into the following equation:

$$\left(\frac{dy}{d\eta}\right)^2=(\alpha_1-1)[(y-y_0)^2-\Delta^2],$$

where

$$y_0=2\mu q_0(\alpha_1-1)^{-1},\quad \Delta^2=\mu(\alpha_1-1)^{-1}+y_0^2. \tag{36}$$

The substitution $y=y_0\pm\Delta\cosh\varphi$ reduces this equation to the trivial one $d\varphi/d\eta=(\alpha_1-1)^{-1/2}$. Thus, we have the solution of Eq. (35) written as

$$f(t,x)=\{y_0\pm\Delta\cosh[\sqrt{(\alpha_1-1)}(\eta-\eta_0)]\}^{-1}. \tag{37}$$

If $q_0\to 0$ this expression reproduces the formula (24). The electric field of the electromagnetic wave is given by the following expression:

$$e(t,x)=e_0+\alpha\{y_0\pm\Delta\cosh[\sqrt{(\alpha_1-1)}(\eta-\eta_0)]\}^{-1}=e_0+\frac{\alpha(\alpha_1-1)}{2\mu q_0\pm\sqrt{\mu(\alpha_1-1)+4\mu^2q_0^2}\cosh[\sqrt{(\alpha_1-1)}(\eta-\eta_0)]}. \tag{38}$$

The plus sign in (38) corresponds to a bright spike of the electromagnetic field on a constant background; the minus sign in (38) corresponds to a dark solitary wave: the narrow hole propagating on background.

Now we can consider the special case when the parameter α_1 is equal unity. Equation (35) then takes the form

$$\left(\frac{df}{d\eta}\right)^2=-4\mu q_0 f^3-\mu f^4. \tag{39}$$

Substitution of $f=-1/y$ transforms it into $(dy/d\eta)^2=\mu(4q_0 y-1)$, the integral of which leads to the following expression for f:

$$f(t,x)=\frac{-4q_0}{1+4\mu q_0^2(x-t/\alpha-x_0)^2}. \tag{40}$$

Here, x_0 is the constant of integration which indicates the location of the maximum of the steady-state dark pulse. One may name this pulse ”algebraic soliton,” due to its decay rate with time and coordinate. For this solution we have the following relations:

$$q(t,x)=q_0-\frac{4q_0}{1+4\mu q_0^2(x-t/\alpha-x_0)^2} \tag{41}$$

and

$$e(t,x)=q_0+2\mu q_0^3-\frac{4q_0(1+6\mu q_0^2)}{1+4\mu q_0^2(x-t/\alpha-x_0)^2}. \tag{42}$$

Note that when the initial medium state q_0 is large, the amplitude of the electric field is much larger than the medium variable. For $q_0\gg 1$ we have a dark solitary wave having a bright spot in the center, while for $q_0\ll -1$ we have a solitary wave superimposed on a nonzero background.

V. STABILITY OF A STEADY-STATE PULSE ON A ZERO BACKGROUND

The problem of the stability of the steady-state solutions of the RMD equations can be considered with variational methods [39]. For that we introduce the following functional:

$$W[q,e] = H + \frac{2}{\alpha} I_2, \tag{43}$$

where the Hamiltonian is taken in the form (12). In the variation of $W[q,e]$ the fields q and e are considered to be independent resulting, in

$$\delta W = \int_{-\infty}^{+\infty} \left[\frac{\partial^2 q}{\partial x^2} + q + 2\mu q^3 - e \right] \delta q dx + \int_{-\infty}^{+\infty} [\alpha^{-1} e - q] \delta t \, dx.$$

Thus, the variational problem $\delta W=0$ leads to the Eqs. (22) and (23). In this section we will use x instead of the variable $\eta = x - t/\alpha$. Multiplying Eq. (23) by q and integrating it over x, we obtain

$$- W_1 - bW_2 + 2\mu W_4 = 0, \tag{44}$$

where $b=\alpha-1$, and

$$W_1 = \int_{-\infty}^{+\infty} \left(\frac{\partial q}{\partial x} \right)^2 dx, \quad W_n = \int_{-\infty}^{+\infty} q^n dx, \quad n \geq 2 \tag{45}$$

Multiplying Eq. (23) by $x\partial_x q$ and integrating it over x, we obtain after some manipulations the other relation between the integrals under consideration:

$$W_1 - bW_2 + \mu W_4 = 0. \tag{46}$$

Combining (44) and (46), we obtain

$$W_1 = bW_2/3, \quad \mu W_4 = 2bW_2/3, \tag{47}$$

and

$$H = - (5\alpha - 2)W_2/6. \tag{48}$$

One can check that the Hamiltonian (13) takes the same value as (48). Thus, the Hamiltonian is negative for $\alpha > 1$. Using formula (47), one can find $W=-bW_2/3<0$ as well.

To solve the problem of stability of steady-state solutions, we can use Lyapunov's theorem which, in the case of boundedness of the Hamiltonian (or any suitable functional) from above or below, gives absolute stability to a solution realizing a maximum or a minimum. Now we prove that the Hamiltonian is bounded from below for $\alpha > 1$.

Let us consider the scaling transformation containing I_2: $q(x) \to \lambda^{1/2} q(\lambda x)$ [39,40], for which the Hamiltonian (13) becomes a function of the parameter λ:

$$\tilde{H}(\lambda) = \frac{1}{2}(W_1\lambda^2 - W_2 - 3\mu W_4 \lambda). \tag{49}$$

When $\lambda \to 1$ we have $\tilde{H}(\lambda) \to H$. An extremum of $\tilde{H}(\lambda)$ is attained at the point

$$\lambda_0 = 3\mu W_4/2W_1. \tag{50}$$

As $d^2\tilde{H}(\lambda)/d\lambda^2 = W_1 > 0$, at this point the function $\tilde{H}(\lambda)$ has a minimum. For $\lambda_0 \neq 1$ we have $|\min \tilde{H}(\lambda)| > |H|$ and for $\lambda_0 = 1$ $|\min \tilde{H}(\lambda)| = |H|$. Hence, $|\min \tilde{H}(\lambda)| \geq |H|$. The substitution of (50) into (49) results in

$$\min \tilde{H}(\lambda) = -\frac{1}{2}\left(W_2 + \frac{9\mu^2 W_4^2}{4W_1} \right).$$

To estimate the integral W_4 we can use the inequality (1.39) from [39]:

$$W_4 \leq 2W_1^{1/2} W_2^{3/2}. \tag{51}$$

Using the inequality

$$|H| \leq |\min \tilde{H}(\lambda)| \leq \frac{1}{2}\left(|W_2| + \left| \frac{9\mu^2 W_4^2}{4W_1} \right| \right),$$

and (51), we obtain

$$|H| \leq \frac{W_2}{2}(1 + 9\mu^2 W_2^2) \tag{52}$$

As the Hamiltonian is bounded from below, the functional W is bounded as well. According to Lyapunov's theorem, this proves stability of the steady-state solutions (24).

To consider the stability of the steady-state solutions with nonzero boundary conditions (38), we should start from Eqs. (33) and find the functional that would be minimized by the solutions of Eq. (34). In this way we could obtain the nonlinearity that is smaller than q^4. This corresponds to the problem discussed in [40]. The principal role in the analysis is due to the integral I_2. The steady-state ESP (38) is also seen to be stable. We will confirm this using a numerical simulation.

VI. NUMERICAL SIMULATION OF THE PROPAGATION AND COLLISIONS OF THE PULSES

The propagation and interaction of the steady-state pulses in the dispersive medium with both quadratic and quadratic-cubic nonlinearities were thoroughly studied in [18,19]. There, direct simulations demonstrated the strong stability of the pulses against various perturbations.

Here we have considered the cubic nonlinear medium and found two types of steady-state pulses: an electromagnetic spike propagating in a ground state medium and a pulse propagating in a medium polarized by a constant electric field. In this last case, the electromagnetic pulse propagates steadily over the constant electric background. In the case of zero background, we obtain a one-parameter family of stable solutions. The perturbations (for example, collisions) can transform one of them into another member of this family. Frequently, a collision results in radiation of quasiharmonic waves. In addition, we cannot rule out the existence of non-steady-state solutions, apart from the ones we found, which are not stable but are long-lived. Numerical simulations may be useful to study this particular aspect.

For this purpose, we write the system (7) in the following form:

$$\frac{\partial e}{\partial t} = - p, \tag{53}$$

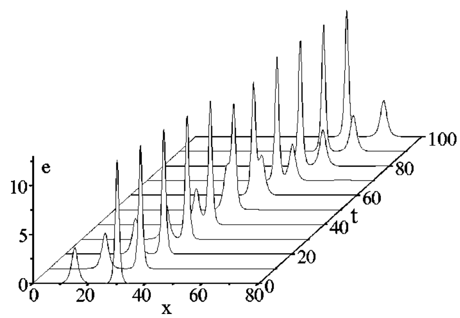

FIG. 1. Collision between two bright steady-state pulses on a zero background. Their relative velocities differ sufficiently and the interaction does not change their shapes. The parameter $\mu=1/3$.

$$\frac{\partial q}{\partial x}=p, \quad \frac{\partial p}{\partial x}=e-q-2\mu q^3. \tag{54}$$

Given initial conditions $q(t=0,x)$, $p(t=0,x)$, and $e(t=0,x)$ and boundary conditions $q(t,x)=q_0$, ($q_0=0$ or $q_0\neq 0$) $\partial q(t,x)/\partial x=0$, for $x\to\pm\infty$, the basic algorithm is the following:

(1) integrate (53) in t to obtain $e(dt,x)$,

(2) integrate (54) in x to obtain $q(dt,x+dx),p(dt,x+dx)$,

(3) go to step 1.

One can use any technique of numerical integration of ordinary differential equations to solve this system both in x and t. We employed the fourth-order Runge-Kutta routine. As initial conditions, we took the analytical solutions given by Eqs. (25) and (38) at $t=0$.

A. Propagation of pulses on a zero background

In this subsection we consider the stability of the steady-state solutions corresponding to spikes propagating in nonpolarized media, where the oscillators are initially in the ground state. In this case the initial and boundary conditions in the numerical simulations have been chosen as (21) with $e_0(x)$ defined by the expression (25). Below, the parameter μ is set to 0.3 unless otherwise indicated. As the velocities of the pulses are determined by their widths, the angle of the trajectories in the (x,t) plane, and therefore the interaction time of the pulses can be modified by choosing their initial widths.

For equal polarities, if the relative velocities of the colliding pulses are considerably different, the pulses do not change their form and velocity after interaction. Figure 1 shows the collision of two bright spike solutions of (25) with $\alpha=2$ and $\alpha=4$. Strictly speaking, the solitary wave solutions of the model under consideration are not solitons since the system of Eqs. (7) has no Painlevé properties [41], but numerical simulations prove that these steady-state pulses collide almost elastically.

When decreasing the relative velocities of the colliding pulses, which are chosen with equal polarities, a strong mutual energy exchange takes place and the two colliding pulses never completely overlap. Figure 2 is a typical picture illustrating this result, which is quite similar to the classical description of collisions between solitons in the modified Korteweg–de Vries equation [38]. Numerical simulations show that the collisions between pulses are almost elastic independently of the polarities as long as the amplitudes are different. This is in contrast to the collision between pulses in the quadratic-cubic nonlinear medium [19]. This difference can be due to the symmetry of the Hamiltonian (13) of the cubic Maxwell-Duffing model, which is different from the case of [18,19].

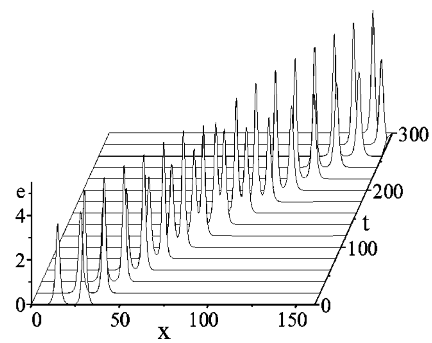

FIG. 2. Energy exchange between two bright pulses traveling at very close velocities ($\alpha=2$ and $\alpha=2.4$).

Increasing the initial pulse energy beyond the energy of the steady-state pulse results in its decay into one or more steady-state pulses (depending on the initial pulse energy) and radiation. The formation of a steady-state pulse from an arbitrary initial (smooth) pulse also illustrates the stability of the solitary waves under consideration.

To summarize this study of the collisions between two spikes, one can say that the result of the collision of the two spikes depends only weakly on the difference of the pulse widths. The two spikes can penetrate through each other without appreciable changes, if their widths are different enough. However, the collision leads to more interesting results when the pulses velocities are close. In this case the energy due to the decay of the smaller pulse (irrespective of its polarity) is transferred to the larger pulse and a small solitary wave appears during the interaction process.

The stability of the pulses under weak perturbations (i.e., a low-amplitude harmonic wave packet) is also very interesting. We find that the steady-state pulses under consideration appear to be stable in this sense. To conclude, our investigation demonstrates that the steady-state solutions of the RMD equations on a zero background behave similarly to solitons of completely integrable models.

B. Pulse propagation on a nonzero background

In the absence of an electromagnetic wave, a medium polarized by a constant electric field is stable. For this model it can be shown that small perturbations of the background are not amplified so that there is no modulational instability.

Now let us consider the stability of the steady-state electromagnetic spikes we found propagating on such a background. The initial and boundary conditions have been chosen as (32), where the initial solitary pulse has the following form:

$$e_{sol}(t=0,x)=e_0+\alpha\{y_0\pm\Delta\cosh[\sqrt{(\alpha_1-1)}(\eta-\eta_0)]\}^{-1}$$

with Δ defined by (36).

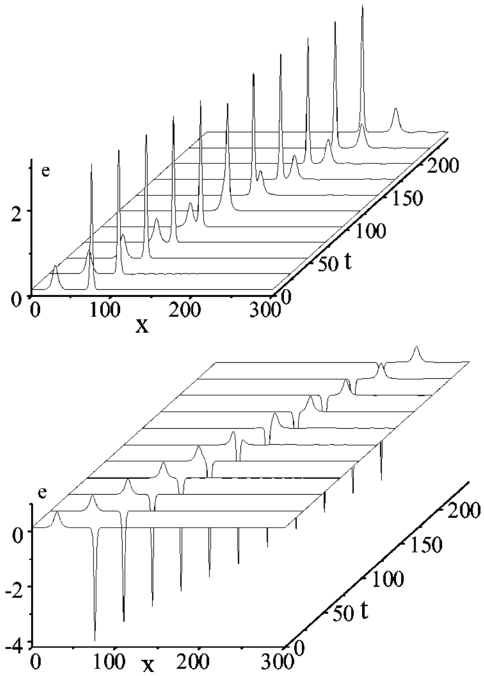

FIG. 3. Interactions between steady-state pulses on a background corresponding to $q_0=0.15$ with different polarities. The faster pulse is such that $\alpha_1=1.2$ and the slower one is such that $\alpha_1=2$.

To investigate the stability of the solitary waves (38) over a background under strong perturbations, collisions of these steady-state pulses were simulated. It was found that pulses with sufficiently different amplitudes interact almost like solitons. This result does not depend on the relative polarity of the colliding pulses, as seen from Fig. 3. However, a weak radiation is emitted after the collision, which is therefore not completely elastic. This emission of radiation decreases when the pulse amplitudes are different, but we did not observe any threshold above which it disappears.

As in the case of a zero background, the perturbation of a steady-state pulse by a weak harmonic wave does not destroy it; at the same time the harmonic wave packet transforms into a dispersive wave. An example is presented in Fig. 4, which shows the evolution of a bright steady-state pulse initially perturbed by a harmonic wave packet, i.e.,

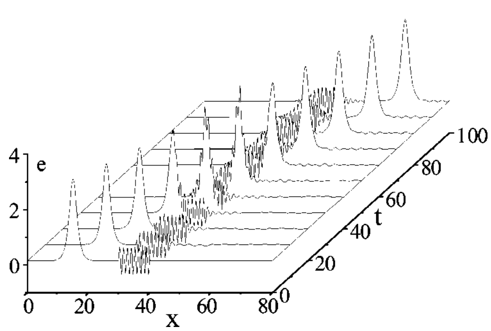

FIG. 4. Evolution of a pulse with $\alpha=2$ in a previously polarized medium characterized by $q_0=0.15$. The pulse is initially perturbed by a harmonic wave packet.

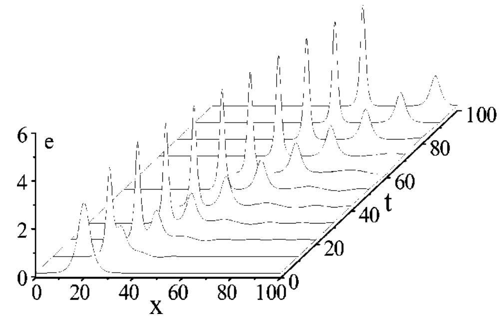

FIG. 5. Formation of steady-state pulses from a high-energy initial pulse in a polarized medium.

$$e(t=0,x) = e_{sol}(t=0,x-15) + 0.5[\vartheta(x-30) - \vartheta(x-40)]\cos 5x$$

where $\vartheta(x)$ is the Heaviside step-like function: $\vartheta(x)=0$ if $x<0$, and $\vartheta(x)=1$ if $x>0$. This steady-state pulse is destroyed when the amplitude of the harmonic wave packet is of the order of the soliton amplitude. The breakup of an initial high-energy pulse into steady-state pulses (Fig. 5) also illustrates the stability and soliton-like behavior of the solitary waves under consideration.

C. Breather-like pulses on a zero background

Equation (10) can be represented in the form of the modified Korteweg–de Vries (mKdV) equation with the additional term

$$\frac{\partial q}{\partial t} + \frac{\partial q}{\partial x} - 6\mu q^2 \frac{\partial q}{\partial x} - \frac{\partial^3 q}{\partial x^3} = R[q], \tag{55}$$

where

$$R[q] = -\frac{\partial^2}{\partial x^2}\left(\frac{\partial q}{\partial t} + \frac{\partial q}{\partial x}\right) - 6\mu q^2 \left(\frac{\partial q}{\partial t} + \frac{\partial q}{\partial x}\right),$$

is the difference between the RMD and mKdV equations. Thus, a good agreement between the solutions of these two models can be expected if $q_{,x} \approx -q_{,t}$. In that case, the first equation of the system (7) gives $e \approx q$ and (10) leads to

$$\frac{\partial e}{\partial t} + \frac{\partial e}{\partial x} - 6\mu e^2 \frac{\partial e}{\partial t} - \frac{\partial^3 e}{\partial x^3} = R[e].$$

The possible occurrence of breather-like pulses in the RMD model was an objective of our numerical investigation. Since the evolution equations of the RMD and mKdV models resemble each other and the numerical simulation of the steady-state pulses of the RMD model shows significant stability both during collisions and perturbations, it seems natural to consider a breather solution of the mKdV equation as an initial condition for the RMD equations, given by

$$e(x,t=0) = -\frac{4\beta}{\alpha_2} \frac{\alpha_2 \cos\theta_2 \cosh\theta_1 - \beta \sinh\theta_1 \sin\theta_2}{\cosh^2\theta_1 + (\beta/\alpha_2)^2 \sin^2\theta_2}, \tag{56}$$

with

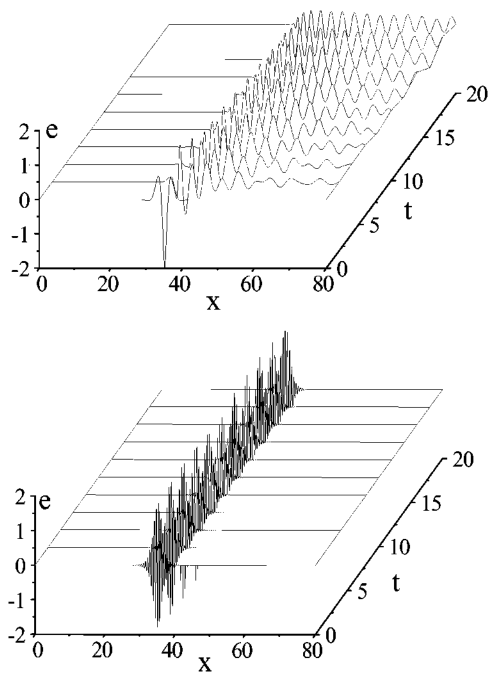

FIG. 6. A breather solution of the mKdV model in the RMD model. (a) corresponds to $\alpha_2 = 0.5$, while (b) corresponds to $\alpha_2 = 1.5$.

$$\theta_1 = 2\beta(x - x_{10}) + 8\beta(\beta^2 - 3\alpha_2^2 - 0.25)t,$$

$$\theta_2 = 2\alpha_1(x - x_{20}) + 8\alpha_2(\alpha_2^2 - 3\beta^2 + 0.25)t.$$

Therefore, we use the breather solution of the mKdV equation (56) as an initial condition for the RMD model and will consider the stability of these pulses depending on their frequency. In the following simulations, $\beta = 0.5$ and α_2 is varied.

As seen from Fig. 6(a), a low-frequency pulse is broadened due to dispersion and decays into quasiperiodic waves. Localized breather-like pulses do not form in this case. Increasing the initial pulse frequency leads to pulse stabilization, as shown in Fig. 6(b). However, we do not find a sharp transition from the dispersion broadening regime to the steady-state propagation. The distance for which distortions become significant increases monotonically with the frequency of the modulation α_2. The collisions of such high-frequency pulses with a steady-state pulse of the RMD model (Fig. 7) also demonstrate the stability of these breather-like pulses even though they are not exact solutions of the model under consideration. In Fig. 8 we show the evolution of a high-frequency pulse obtained by modulation of a steady-state pulse of the RMD model by a harmonic wave. Here the distortions of an initial pulse are most pronounced for $\alpha_2 \approx 1$, i.e., near the eigenfrequency of the oscillators (3). Increasing the initial pulse frequency leads to the stabilization of the modulated pulse envelope.

As it appears, both a high-frequency mKdV breather and a high-frequency pulse obtained by the modulation of a steady-state pulse of the RMD model by a harmonic wave propagate steadily. The collisions of a steady-state pulse with

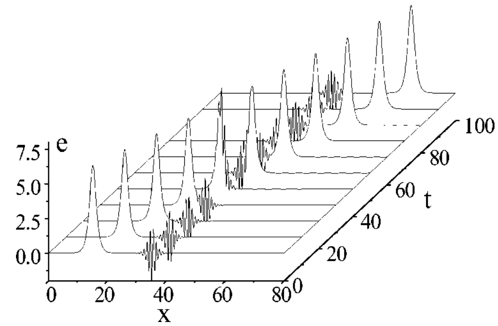

FIG. 7. Collision between a steady-state pulse of the RMD model and a high-frequency pulse corresponding to a mKdV breather, $\alpha_2 = 3$. The steady-state pulse is characterized by $\alpha = 3$. The parameter $\mu = 0.1$.

these high-frequency pulses also demonstrate the robustness of such quasibreathers. The spectrum of the pulse located near the frequency of the carrier wave is unchanged as the pulse propagates. By using the multiple scale expansion method [38], we found that the envelope of the high-frequency modulated electromagnetic pulse evolves according to the nonlinear Schrödinger equation. This explains the long life of the high-frequency breather-like pulse of the RMD model. We can then conclude that high-frequency breather-like pulses in the RMD model are very close to the genuine breathers of a completely integrable system.

VII. CONCLUSION

We have analyzed a model for the propagation of extremely short unipolar pulses of an electromagnetic field in a medium represented by anharmonic oscillators with a cubic nonlinearity. The model takes into account the dispersion properties (in the linear limit) and the nonlinear response of the medium. This is the simplest generalization of the well-known Lorenz model used to describe linear optical properties in condensed matter. The cubic nonlinearity is the simplest anharmonic correction to the Lorenz model and it results in the Duffing oscillator.

Before commenting further on our results, let us find the region of applicability of the model that we considered. For that, one needs to estimate the amplitude of the steady-state solitary wave. In the general case, the representation of the potential $U(X)$ near the equilibrium position as a power series in the displacement X of an electron from its equilibrium position gives the following approximations for the anharmonicity coefficients:

$$|\kappa_n| = \frac{1}{(n-1)!m}\left(\frac{d^n U}{dX^n}\right)_0 \approx \frac{1}{(n-1)!m}\left(\frac{U_0}{r_0^n}\right). \qquad (57)$$

Hence,

$$\omega_0^2 = \frac{U_0}{mr_0^2}, \quad |\kappa_3| = \frac{U_0}{6mr_0^4}, \qquad (58)$$

where r_0 corresponds to the equilibrium position and U_0 is the ionization potential. For comparison, if we consider a Morse potential, then we obtain $\omega_0^2 = 2U_0/mr_0^2$ and κ_3

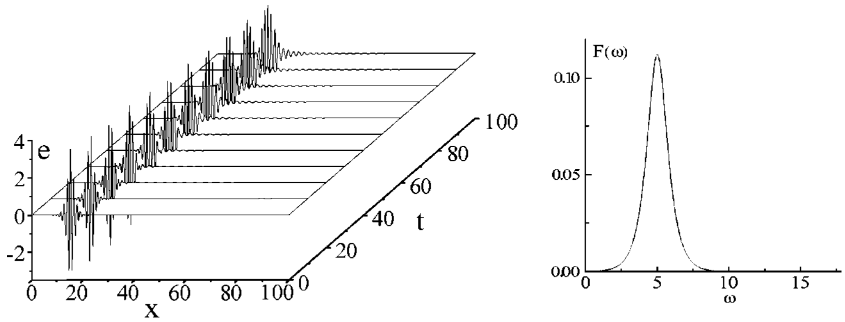

FIG. 8. Left panel: evolution of the initial pulse $e(t=0,x)=\alpha\sqrt{(\alpha-1)/\mu}\,\mathrm{sech}[\sqrt{(\alpha-1)}(x-x_0)]\cos[5(x-x_0)]$ with $\alpha=2$. Right panel: Fourier spectrum of the pulse for the initial condition. It does not change with time. The parameter $\mu=0.3$.

$=7U_0/3mr_0^4$ [19]. By using the normalizing parameters from (5)—$A_0=m\omega_0^2 r_0/\mathfrak{e}$, $X_0=r_0$, and $2\mu=|\kappa_3|r_0^3/\omega_0^2$—we can estimate the peak of the amplitudes $E_m=\max E(x,t)$ and $X_m=\max X(x,t)$ for a steady-state pulse, as

$$E_m=\alpha(\alpha-1)^{1/2}A_{\mathrm{at}},\quad X_m=(\alpha-1)^{1/2}r_0, \tag{59}$$

where $A_{at}=U_0/\mathfrak{e}r_0$ is the strength of the atomic field. The peak amplitude of the steady-state pulse must not exceed A_{at}. According to [17] for xenon, $A_{at}\approx 2\times 10^9$ V/cm. Furthermore, the anharmonic-oscillator model assumes that the magnitude of q is less than 1, and this leads to the inequality $\alpha-1<1$.

When the strength of the electric field is extremely high, we can neglect the effect of the atomic potential and consider the electron as free in this field. This allows to estimate the increase in kinetic energy due to the field as

$$W_{kin}\approx\frac{(eE_m t_p)^2}{2m}, \tag{60}$$

where t_p is the pulse duration. Ionization does not happen if $W_{kin}<U_0$. Let us use the expression for the mass from the formula $\omega_0^2=2U_0/mr_0^2$. This leads to the no-ionization condition

$$eE_m\omega_0 t_p<2U_0/r_0. \tag{61}$$

This inequality is equivalent to the inequality $\alpha-1<1$ for the steady-state pulse considered. However, in order to obtain clear illustrations of these effects in the numerical results, the parameter α has been taken both from the interval $1.01<\alpha<1.1$ and from the interval $2<\alpha<5$. The unidirectional wave approximation is valid if $\omega_p/\omega_0\ll 1$. This limits the density of atoms. Typical concentrations of gas or impurities in glass satisfy this condition so that we can reduce the wave equation to a first-order evolution equation. Therefore, the solutions that we have found could be observed by sending a short- and large-amplitude laser pulse in these materials.

In this article, we have considered the reduced Maxwell-Duffing model in detail. The Lagrangian density of the RMD model was considered and three integrals of motion were found. Two families of exact analytical solutions with positive and negative polarities, have been found as moving solitary pulses. The first kind of steady-state ESP is an electromagnetic spike propagating in a nonlinear medium. This was discussed earlier in [16,17,21]. Here we also obtained this solution by an alternative method as a soliton of the bilinear form of the RMD equations. Using the integrals of motion and a variational method we show that these solitons are stable. This steady-state ESP is an electromagnetic spike propagating on a nonzero electric background. These can be both bright and dark ESPs. Contrary to the ESP on a zero background, here pulses of different polarities have different amplitudes. The stability of these solutions can also be proved by the variational method. We investigated numerically the propagation of both kinds of ESPs and demonstrated the stability of steady-state solutions of RMD equations both on a zero background and a nonzero one. In both cases we found that inelastic effects are not essential. Of course, there are some perturbations [i.e., high-frequency modulation with large amplitude, or low-frequency modulation just as Fig. 6(b) demonstrates] that can destroy the solitons. Finally, we do not observe the formation of a bound state of steady-state pulses. This does not seem possible.

By considering the evolution of modulated initial pulses, we found that there are long-lived high-frequency breather-like solutions of the RMD equations. The envelope of these pulses can be described approximately by the nonlinear Schrödinger equation. This slowly varying envelope approximation is valid when the frequency of modulation is much greater than the eigenfrequency of the Duffing oscillators. In that regime the breather-like pulses are robust.

More generally, the reduced Maxwell-Duffing system of equations does not seem to be integrable contrary to its relative, the reduced Maxwell-Bloch system. It does, however, preserve some of the regular features of the RMB system, as evidenced by the limit of elastic collisions between pulses and the existence of robust high-frequency breathers.

ACKNOWLEDGMENTS

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APPENDIX: HIGH-FREQUENCY LIMIT OF RMD EQUATIONS

Let us consider Eqs. (10)

$$\frac{\partial q}{\partial t}+\frac{\partial q}{\partial x}+6\mu q^2\frac{\partial q}{\partial t}+\frac{\partial^3 q}{\partial t\partial x^2}=0. \tag{62}$$

Following a multiscale expansion, we write the variable q as the series

$$q = \epsilon(q_0 + \epsilon q_1 + \epsilon^2 q_2 + \epsilon^3 q_3 + \cdots), \tag{63}$$

where the derivatives are represented as

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial t_0} + \epsilon\frac{\partial}{\partial t_1} + \epsilon^2\frac{\partial}{\partial t_2} + \cdots;$$

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial x_0} + \epsilon\frac{\partial}{\partial x_1} + \epsilon^2\frac{\partial}{\partial x_2} + \cdots. \tag{64}$$

Substituting (63) into (62) and collecting the terms with equal-order results in the following equations:

$$\left(\frac{\partial}{\partial x_0} + \frac{\partial}{\partial t_0} + \frac{\partial^3}{\partial t_0 \partial x_0^2}\right)q_0 = 0, \tag{65}$$

$$\left(\frac{\partial}{\partial x_0} + \frac{\partial}{\partial t_0} + \frac{\partial^3}{\partial t_0 \partial x_0^2}\right)q_1 + \left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial t_1}\right)q_0 + 2\frac{\partial^3 q_0}{\partial t_0 \partial x_0 \partial x_1} + \frac{\partial^3 q_0}{\partial t_1 \partial x_0^2} = 0, \tag{66}$$

$$\left(\frac{\partial}{\partial x_0} + \frac{\partial}{\partial t_0} + \frac{\partial^3}{\partial t_0 \partial x_0^2}\right)q_2 + \left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial t_1}\right)q_1 + \left(\frac{\partial}{\partial x_2} + \frac{\partial}{\partial t_2}\right)q_0 + \frac{\partial^3 q_0}{\partial t_0 \partial x_1^2} + \frac{\partial^3 q_0}{\partial t_2 \partial x_0^2} + \frac{\partial^3 q_1}{\partial t_1 \partial x_0^2} + 2\frac{\partial^3 q_0}{\partial t_1 \partial x_0 \partial x_1} + 2\frac{\partial^3 q_0}{\partial t_0 \partial x_0 \partial x_2} + 2\frac{\partial^3 q_1}{\partial t_0 \partial x_0 \partial x_1} + 6\mu q_0^2 \frac{\partial q_0}{\partial t_0} = 0, \tag{67}$$

and so on. The solution of the first equation is

$$q_0 = a(x_1, t_1, x_2, t_2, \dots)\exp\{ikx_0 - i\omega t_0\} + \text{c.c.}, \tag{68}$$

where the frequency and wave number are connected by the dispersion relation

$$\omega = k(1-k^2)^{-1}.$$

This is the dispersion relation for the linearized RMD equations.

The substitution of (68) into (53) yields the equation

$$\left(\frac{\partial}{\partial x_0} + \frac{\partial}{\partial t_0} + \frac{\partial^3}{\partial t_0 \partial x_0^2}\right)q_1 = e^{i\theta}\left((k^2-1)\frac{\partial a}{\partial t_1} - (1+2k\omega)\frac{\partial a}{\partial x_1}\right) + e^{-i\theta}\left((k^2-1)\frac{\partial a^*}{\partial t_1} - (1+2k\omega)\frac{\partial a^*}{\partial x_1}\right). \tag{69}$$

Here, $\theta = kx_0 - \omega t_0$. The group velocity is

$$v_g = d\omega/dk = (1+k^2)/(1-k^2)^2.$$

From the dispersion relation we get

$$1 + 2k\omega = (1+k^2)/(1-k^2),$$

thus,

$$(k^2-1)\frac{\partial a}{\partial t_1} - (1+2k\omega)\frac{\partial a}{\partial x_1} = (k^2-1)\left(\frac{\partial a}{\partial t_1} + v_g\frac{\partial a}{\partial x_1}\right).$$

If we choose $a = a(t_1 - x_1/v_g, t_2, x_2, \dots)$, then the terms on the right side of Eq. (69) are equal to zero. Let q_1 be zero. The substitution of (68) into (67) taking $q_1 = 0$ into account then results in

$$\left(\frac{\partial}{\partial x_0} + \frac{\partial}{\partial t_0} + \frac{\partial^3}{\partial t_0 \partial x_0^2}\right)q_2 = 6i\mu\omega a^3 \exp\{3i\theta\} - 6i\mu\omega a^{*3}\exp\{-3i\theta\} + e^{i\theta}\left((k^2-1)\frac{\partial a}{\partial t_2} - (1+2k\omega)\frac{\partial a}{\partial x_2} + i\frac{\omega + 2kv_g}{v_g^2}\frac{\partial^2 a}{\partial \xi} + 6i\mu\omega|a|^2 a\right) + e^{-i\theta}\left((k^2-1)\frac{\partial a^*}{\partial t_2} - (1+2k\omega)\frac{\partial a^*}{\partial x_2} - i\frac{\omega + 2kv_g}{v_g^2}\frac{\partial^2 a^*}{\partial \xi} - 6i\mu\omega|a|^2 a^*\right).$$

Here we use the new variable $\xi = t_1 - x_1/v_g$. As the complex envelope a does not depend on t_0 and x_0, the function q_2 will be a superposition of $\sin 3\theta$ and $\cos 3\theta$ only if the expression in the brackets on the right-hand side of this equation is zero. Thus, for the complex envelope in (68) we get the equation

$$(k^2-1)\frac{\partial a}{\partial t_2} - (1+2k\omega)\frac{\partial a}{\partial x_2} + i\frac{\omega + 2kv_g}{v_g^2}\frac{\partial^2 a}{\partial \xi} + 6i\mu\omega|a|^2 a = 0. \tag{70}$$

By assuming that a does not depend on x_2, this equation can be transformed into the standard form

$$i(1-k^2)\frac{\partial a}{\partial t_2} + \frac{\omega + 2kv_g}{v_g^2}\frac{\partial^2 a}{\partial \xi} + 6\mu\omega|a|^2 a = 0. \tag{71}$$

To conclude we have shown that the envelope of a high-frequency modulated electromagnetic pulse evolves according to the nonlinear Schrödinger equation. This can explain the long lifetime of the high-frequency breather-like pulse of the RMD model.

[1] R. K. Bullough and F. Ahmad, Phys. Rev. Lett. **27**, 330 (1971).
[2] J. C. Eilbeck and R. K. Bullough, J. Phys. A **5**, 820 (1972).
[3] J. D. Gibbon, P. J. Caudrey, R. K. Bullough, and J. C. Eilbeck, Lett. Nuovo Cimento Soc. Ital. Fis. **8**, 775 (1973).
[4] J. C. Eilbeck, J. D. Gibbon, P. J. Caudrey, and R. K. Bullough, J. Phys. A **6**, 1337 (1973).
[5] R. K. Bullough, P. J. Caudrey, J. C. Eilbeck, and J. D. Gibbon, Opto-electronics (London) **6**, 121 (1974).
[6] R. K. Bullough, P. M. Jack, P. W. Kitchenside, and R. Saunders, Phys. Scr. **20**, 364 (1979).
[7] R. K. Bullough, J. Mod. Opt. **48**, 2029 (2000).
[8] A. I. Maimistov, Quantum Electron. **27**, 935 (1997).
[9] L. W. Casperson, Phys. Rev. A **57**, 609 (1998).
[10] M. Agrotis, N. M. Ercolani, S. A. Glasgow, and J. V. Moloney, Physica D **138**, 134 (2000).
[11] M. A. Agrotis, Phys. Lett. A **315**, 81 (2003).
[12] A. I. Maimistov and J.-G. Caputo, Opt. Spectrosc. **94**, 245 (2003).
[13] V. P. Kalosha and J. Herrmann, Phys. Rev. Lett. **83**, 544 (1999).
[14] X. Song, Sh. Gong, W. Yang, and Zh. Xu, Phys. Rev. A **70**, 013817 (2004).
[15] J. Cheng and J. Zhou, Phys. Rev. A **67**, 041404(R) (2003).
[16] A. E. Kaplan and P. L. Shkolnikov, Phys. Rev. Lett. **75**, 2316 (1995).
[17] A. E. Kaplan, S. F. Straub, and P. L. Shkolnikov, J. Opt. Soc. Am. B **14**, 3013 (1997).
[18] E. V. Kazantseva and A. I. Maimistov, Phys. Lett. A **263**, 434 (1999).
[19] E. V. Kazantseva, A. I. Maimistov, and B. A. Malomed, Opt. Commun. **188**, 195 (2001).
[20] S. A. Kozlov and S. V. Sazonov, JETP **84**, 221 (1997).
[21] A. I. Maimistov and S. O. Elyutin, J. Mod. Opt. **39**, 2201 (1992).
[22] M. P. Sorensen, M. Brio, G. M. Webb, and J. V. Moloney, Physica D **170**, 287 (2002).
[23] A. E. Kaplan, Phys. Rev. Lett. **73**, 1243 (1994).
[24] A. I. Maimistov, Opt. Spectrosc. **94**, 251 (2003).
[25] K. Akimoto, J. Phys. Soc. Jpn. **65**, 2020 (1996).
[26] Th. Brabec and F. Krausz, Rev. Mod. Phys. **72**, 545 (2000).
[27] A. I. Maimistov, Quantum Electron. **30**, 287 (2000).
[28] R. K. Bullough, *Optical Solitons: Twenty Seven Years of the Last Millennium and Three More Years of the New?*, in Mathematics and the 21st Century, edited by A. A. Ashour and A.-S. Obada (World Scientific, Singapore, 2001) pp. 69–121.
[29] S. O. Elyutin and A. I. Maimistov, Chaos, Solitons Fractals **8**, 369 (1997).
[30] C. R. Stroud, Jr., C. M. Bowden, and L. Allen, Opt. Commun. **67**, 387 (1988).
[31] C. M. Bowden, A. Postan, and R. Inguva, J. Opt. Soc. Am. B **8**, 1081 (1991).
[32] Ch. M. Bowden and J. P. Dowling. Phys. Rev. A **47**, 1247 (1993).
[33] A. A. Afanasev, R. A. Vlasov, O. K. Khasanov, T. V. Smirnova, and O. M. Fedotova, J. Opt. Soc. Am. B **19**, 911 (2002).
[34] N. Bloembergen, *Nonlinear Optics* (Benjamin, New York, 1965).
[35] P. N. Butcher and D. Cotter, *The Elements of Nonlinear Optics* (Cambridge University Press, Cambridge, U.K., 1990).
[36] C. A. Hurst, *Recent Developments in Mathematical Physics*, edited by H. Mitter and L. Pittner (Springer-Verlag, Berlin, 1987), pp. 18–52.
[37] R. Hirota and J. Satsuma, Suppl. Prog. Theor. Phys. **59**, 64 (1976).
[38] M. J. Ablowitz and H. Segur, *Solitons and the Inverse Scattering Transform* (SIAM, Philadelphia, 1981).
[39] E. A. Kuznetsov, A. M. Rubenchik, and V. E. Zakharov, Phys. Rep. **142**, 103 (1986).
[40] E. A. Kuznetsov, Phys. Lett. **101**, 314 (1984).
[41] Nikolai A. Kudryashov (private communication).